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Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition

Eduardo Cerpa and Jean-Michel Coron

Abstract

This paper deals with the stabilization problem for the Korteweg-de Vries equation posed on a bounded interval. The control acts on the left Dirichlet boundary condition. At the right end-point, Dirichlet and Neumann homogeneous boundary conditions are considered. The proposed feedback law forces the exponential decay of the system under a smallness condition on the initial data. Moreover, the decay rate can be tuned to be as large as desired. The feedback control law is designed by using the backstepping method.

Key words. Korteweg-de Vries equation, stabilization by feedback, backstepping

AMS subject classifications. 35Q53, 93D15

I. Introduction

The Korteweg-de Vries (KdV) equation

$$u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0, \tag{1}$$

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posed on a bounded domain $[0, L]$ can be seen as a nonlinear control system where the inputs are the boundary data. From the nature of this equation, one boundary condition at the left end-point and two boundary conditions at the right end-point have to be imposed. The most studied case considers boundary conditions on

$$u(t, 0), \quad u(t, L), \quad \text{and} \quad u_x(t, L). \tag{2}$$

Surprisingly, the control properties of this system are very different depending on where the controls are located. If we act on the left Dirichlet boundary condition and homogeneous data is considered at the right, then the system behaves like a heat equation and only null-controllability can be proven [22], [7]. On the other hand, if we act on the two right data and homogeneous boundary condition is considered at the left, then the system behaves like a wave equation with an infinite speed of propagation, in the sense that exact controllability holds for any time of control [21]. Another fascinating phenomena occurs when we put only one control input at the right end-point and keep homogeneous the other two boundary conditions: there exist some spatial domains (intervals of some given lengths) for which the corresponding linearized KdV equation is not any more controllable [21], [8]. In despite of that, in these *critical* cases the nonlinearity gives the exact controllability of the nonlinear KdV equation [5], [1], [2]. However, for some critical intervals the applied method requires a minimal time of control, which is not known to be really necessary.

Due to the existence of these critical intervals for the linear system, we do not know if the energy of the KdV equation is decreasing when the three boundary conditions considered are homogeneous (free control case). For instance, if $L = 2\pi$, the time-independent function given by $(1 - \cos(x))$ satisfies

$$u_t(t, x) + u_x(t, x) + u_{xxx}(t, x) = 0, \quad u(t, 0) = u(t, 2\pi) = u_x(t, 2\pi) = 0. \tag{3}$$

Therefore we find out a stationary solution of the linear KdV equation with homogeneous boundary conditions and consequently a solution with non-decreasing energy. This implies that for the free control case with $L = 2\pi$, the linearized system associated to the KdV equation (1) is not exponentially stable.

With a feedback control law acting at the left hand side such phenomenon does not appear and the method we propose in this case allows to address the problem of rapid exponential stabilization: given a desired decay rate, we find a feedback law exponentially stabilizing the system at that rate.

Based on the *hyperbolic* nature of the KdV equation controlled from the right, a method introduced in [10], [30] is used in [3] to get the rapid stabilization of the linear KdV equation from the Neumann boundary condition on the right. This gramian-based approach comes from the finite dimensional theory [18], [9] and was first introduced for PDE in the context of internal control [25]. This method requires the controllability of the linear system and therefore a non critical interval has to be considered.

In this paper, we applied the backstepping method to design the feedback control law. The backstepping method is very known as a ODE control method (see [11] and [4, Section 12.5]). The first extensions to PDE have appeared in [6] and [17]. Later on, Krstic and his collaborators introduced a modification of the method by means of an integral transformation of the PDE. This invertible transformation map the original PDE into an asymptotically stable one. In this context, the first continuous backstepping designs were proposed for the heat equation [16], [27]. The applications to wave equation appeared later in [12], [29], [26]. An excellent starting point to get inside this method is the book [13] by Krstic and Smyshlyaev.

This paper is organized as follows. In Section II, we formulate the problem and state the main result. Section III contains the backstepping design of the feedback control law. Section IV is devoted to prove the exponential decay of the L^2-norm of the solutions for the linearized system around the origin. In Section V, we prove that this result still holds for the nonlinear KdV control system when the initial condition is small enough. Some extensions to the non-constant coefficient case and different boundary conditions are considered in Section VI. Finally, some final remarks are given in Section VII.

Remark 1: The stabilization of the KdV equation by using an internal feedback control law was addressed in [20], [19] (see also [23], [15] for KdV equations with other nonlinearities). They have proved the following semi-global stabilization. Let $L > 0$, $R > 0$ and $a = a(x)$ a damping term satisfying $a(x) \geq a_0 > 0$, forall $x \in \omega$ where ω is nonempty open subset of $(0, L)$. Then, there exist $C = C(R) > 0$ and $\mu = \mu(R) > 0$ such that

$$\|u(t,x)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|u(0,\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0 \tag{4}$$

for any solution of

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + a(x)u(t,x) + u(t,x)u_x(t,x) = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \end{cases} \tag{5}$$

with $\|u(0,\cdot)\|_{L^2(0,L)} \leq R$. This result is different from ours in the sense that the exponential decay rate can not be imposed as large as desired. A key role in their design is played by the damping term $a = a(x)$, which prevents the existence of critical domains and allows to work with a dissipative system for any L. Other internal feedback control laws (static or time-varying ones) for the KdV equation with periodic boundary conditions can be found in [24], [14].

II. Problem Statement and Main Result

Given $L > 0$, we consider the following nonlinear control system on the interval $[0, L]$

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0, \\ u(t,0) = \kappa(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{cases} \tag{6}$$

For any positive λ, we address the problem of building some feedback control law $\kappa(t) = \kappa(u(t,\cdot))$ such that the origin is exponentially stable for the corresponding closed-loop system (6) and the exponential decay rate is λ.

By using the backstepping method, we are able to find such a control law. The design is based on the linearized system around the origin. The linear closed-loop system is exponentially stable and the same result is obtained for the nonlinear KdV equation by adding a smallness condition on the initial data.

Our main theorem is the following.

Theorem 2: For any $\lambda > 0$, there exist a feedback control law $\kappa = \kappa(u(t,\cdot))$, $r > 0$ and $D > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\lambda t}\|u(0,\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0, \tag{7}$$

for any solution of (6) satisfiying $\|u(0,\cdot)\|_{L^2(0,L)} \leq r$.

The feedback law κ is explicitly defined as follows

$$\kappa(t) = \int_0^L k(0,y)u(t,y)dy, \tag{8}$$

where the function $k = k(x,y)$ is characterized in Section III as the solution of a given partial differential equation depending on λ.

Remark 3: As we shall see in Section VI, Theorem 2 also holds if system (6) is replaced by

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) = 0, \\ u(t,0) = \kappa(t), \quad u_x(t,L) = 0, \quad u_{xx}(t,L) = 0. \end{cases} \tag{9}$$

or

$$\begin{cases} u_t(t,x) + \gamma(x)u_x(t,x) + u_{xxx}(t,x) + u(t,x)u_x(t,x) + \delta(x)u(t,x) = 0, \\ u(t,0) = \kappa(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{cases} \tag{10}$$

where γ, δ are given functions in $C^1([0,L])$ and $C([0,L])$, respectively.

III. Control Design

The backstepping method applied here is based on the linear part of the equation. In this way, we consider the control system linearized around the origin

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) = 0, \\ u(t,0) = \kappa(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{cases} \tag{11}$$

Given a positive parameter λ, we look for a transformation $\Pi : L^2(0,L) \to L^2(0,L)$ defined by

$$w(x)=\Pi(u(x)):=u(x)-\int_x^L k(x,y)u(y)dy, \tag{12}$$

such that the trajectory $u=u(t,x)$, solution of (11) with

$$\kappa(t)=\int_0^L k(0,y)u(t,y)dy, \tag{13}$$

is map into the trajectory $w=w(t,x)$, solution of the linear system

$$\begin{cases} w_t(t,x)+w_x(t,x)+w_{xxx}(t,x)+\lambda w(t,x)=0, \\ w(t,0)=0, \quad w(t,L)=0, \quad w_x(t,L)=0. \end{cases} \tag{14}$$

For system (14), called the target system, we have for any $t\geq 0$

$$\frac{d}{dt}\int_0^L |w(t,x)|^2dx=-|w_x(t,0)|^2-2\lambda\int_0^L |w(t,x)|^2dx \leq -2\lambda\int_0^L |w(t,x)|^2dx \tag{15}$$

and therefore we easily obtain for $w=w(t,x)$ the exponential decay at rate λ

$$\|w(t,\cdot)\|_{L^2(0,L)}\leq e^{-\lambda t}\|w(0,\cdot)\|_{L^2(0,L)}, \quad \forall t\geq 0. \tag{16}$$

In Section IV, we prove that, thanks to the invertibility of the map Π, the exponential decay (16) also holds for system (11).

Naturally, we can wonder if this decay rate is sharp. Let us notice that the eigenvalues of system (14) are the eigenvalues of

$$\begin{cases} w_t(t,x)+w_x(t,x)+w_{xxx}(t,x)=0, \\ w(t,0)=0, \quad w(t,L)=0, \quad w_x(t,L)=0, \end{cases} \tag{17}$$

shifted to the left λ units. Thus, we are lead to study the eigenvalues σ of

$$\begin{cases} -\phi'(x)-\phi'''(x)=\sigma\phi(x), \\ \phi(0)=0, \quad \phi(L)=0, \quad \phi'(L)=0. \end{cases} \tag{18}$$

Surprisingly, the location of the eigenvalues of (18) depends on the length of the interval.

From [21], we know that there exist some eigenvalues located on the imaginary axis if and only if $L \in \{2\pi\sqrt{\frac{k^2+k\ell+\ell^2}{3}}/(k,\ell)\in\mathbb{N}^2\}$, which is called the set of critical lengths for this problem. In Figure 1, the first five eigenvalues of system (17) are plot for different values of L. In case (a), $L=1$ (non-critical) and the first eigenvalue σ_1 is approximately -72. The system behaves like a dissipative one. In (b), $L=2\pi$ (critical) and we have $\sigma_1=0$. The system has one conservative component given by the eigenfunction $\phi(x)=1-\cos(x)$. In (c), $L=2\pi\sqrt{7/3}$ and the first two eigenvalues are imaginary numbers $\sigma_1=0.2i$ and $\sigma_2=-0.2i$. This examples show the different behaviors system (14) can have and the important role played by the parameter λ in our design. In conclusion, the decay in (16) is optimal for some values of L.

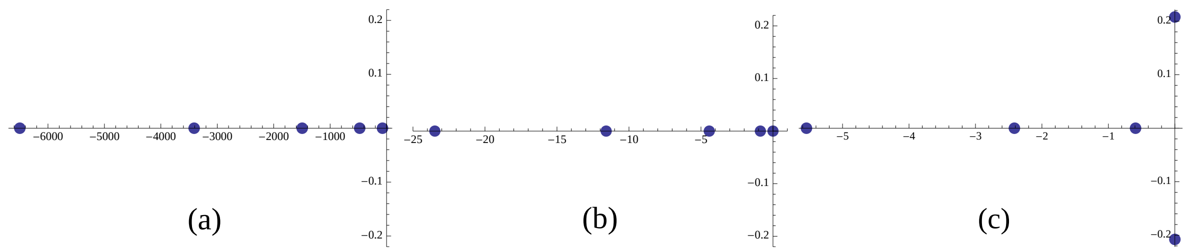

Figure 1. The first five eigenvalues of system (17) for (a) $L=1$, (b) $L=2\pi$, and (c) $L=2\pi\sqrt{7/3}$, respectively.

Let us focus in the key step, which is finding the kernel $k=k(x,y)$ such that $w(t,x)=\Pi(u(t,x))$ satisfies (14). For that, we perform the following computations

$$\begin{aligned} w_t(t,x) &= u_t(t,x)-\int_x^L u_t(t,y)k(x,y)dy \\ &= u_t(t,x)+\int_x^L\left(u_y(t,y)+u_{yyy}(t,y)\right)k(x,y)dy \\ &= u_t(t,x)-\int_x^L u(t,y)\left(k_y(x,y)+k_{yyy}(x,y)\right)dy \\ &\quad -k(x,x)(u(t,x)+u_{xx}(t,x)) \\ &\quad +k_y(x,x)u_x(t,x)-k_{yy}(x,x)u(t,x)+k(x,L)u(t,L) \\ &\quad +k(x,L)u_{xx}(t,L)-k_y(x,L)u_x(t,L)+k_{yy}(x,L)u(t,L), \end{aligned} \tag{19}$$

$$w_x(t,x) = u_x(t,x)+k(x,x)u(t,x)-\int_x^L k_x(x,y)u(t,y)dy, \tag{20}$$

$$
\begin{aligned}
w_{xx}(t,x) &= u_{xx}(t,x) + u(t,x)\tfrac{d}{dx}k(x,x) + k(x,x)u_x(t,x) \\
&\quad + k_x(x,x)u(t,x) - \textstyle\int_x^L k_{xx}(x,y)u(t,y)dy,
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
w_{xxx}(t,x) &= u_{xxx}(t,x) + u(t,x)\tfrac{d^2}{dx^2}k(x,x) + 2u_x(t,x)\tfrac{d}{dx}k(x,x) \\
&\quad + k(x,x)u_{xx}(t,x) + u(t,x)\tfrac{d}{dx}k_x(x,x) + k_x(x,x)u_x(t,x) \\
&\quad + k_{xx}(x,x)u(t,x) - \textstyle\int_x^L k_{xxx}(x,y)u(t,y)dy.
\end{aligned} \tag{22}
$$

Thus, given $\lambda \in \mathbb{R}$ and using (11), we have

$$
\begin{aligned}
&w_t(t,x) + w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) = \\
&\qquad - \int_x^L u(t,y)\Big(k_{xxx}(x,y) + k_x(x,y) + k_{yyy}(x,y) + k_y(x,y) + \lambda k(x,y)\Big)dy \\
&\qquad\quad + k(x,L)u_{xx}(t,L) + u_x(t,x)\Big(k_y(x,x) + k_x(x,x) + 2\frac{d}{dx}k(x,x)\Big) \\
&\qquad\qquad + u(t,x)\Big(\lambda + k_{xx}(x,x) - k_{yy}(x,x) + \frac{d}{dx}k_x(x,x) + \frac{d^2}{dx^2}k(x,x)\Big).
\end{aligned} \tag{23}
$$

After the above computations and since $\frac{d}{dx}k(x,x) = k_x(x,y) + k_y(x,y)$, we obtain that one has (14) for every u solution of (11) with (13) if the kernel $k = k(x,y)$ defined in the triangle $\mathcal{T} := \{(x,y) \,/\, x \in [0,L],\, y \in [x,L]\}$ satisfies

$$
\begin{cases}
k_{xxx}(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_y(x,y) = -\lambda k(x,y), & \text{in } \mathcal{T}, \\
k(x,L) = 0, & \text{in } [0,L], \\
k(x,x) = 0, & \text{in } [0,L], \\
k_x(x,x) = \dfrac{\lambda}{3}(L-x), & \text{in } [0,L].
\end{cases} \tag{24}
$$

Let us make the following change of variable

$$
t = y - x, \quad s = x + y, \tag{25}
$$

and define $G(s,t) := k(x,y)$. We have $k(x,y) = G(x+y, y-x)$ and therefore

$$k_x = G_s - G_t, \quad k_y = G_s + G_t, \tag{26}$$

$$k_{xx} = G_{ss} - 2G_{st} + G_{tt}, \quad k_{yy} = G_{ss} + 2G_{st} + G_{tt}, \tag{27}$$

$$k_{xxx} = G_{sss} - 3G_{sst} + 3G_{stt} - G_{ttt}, \tag{28}$$

$$k_{yyy} = G_{sss} + 3G_{sst} + 3G_{stt} + G_{ttt}. \tag{29}$$

Now, the function $G = G(s,t)$, defined in $\mathcal{T}_0 := \{(s,t) \,/\, t \in [0,L], s \in [t, 2L-t]\}$, satisfies

$$\begin{cases} 6G_{tts}(s,t) + 2G_{sss}(s,t) + 2G_s(s,t) &= -\lambda G(s,t), & \text{in } \mathcal{T}_0, \\ G(s, 2L-s) &= 0, & \text{in } [L, 2L], \\ G(s,0) &= 0, & \text{in } [0, 2L], \\ G_t(s,0) &= \dfrac{\lambda}{6}(s-2L), & \text{in } [0,2L]. \end{cases} \tag{30}$$

Let us transform this system into an integral one. We write the equation in variables (η, ξ), integrate ξ in $(0,\tau)$ and use that $6G_{ts}(\eta, 0) = \lambda$. Next, we integrate τ in $(0,t)$ and use that $G_s(\eta, 0) = 0$. Finally, we integrate η in $(s, 2L-t)$ and use that $G(2L-t,t) = 0$. Thus, we can write the following integral form for $G = G(s,t)$

$$\begin{aligned} G(s,t) = &-\frac{\lambda t}{6}(2L-t-s) \\ &+ \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \lambda G(\eta,\xi)\Big) d\xi d\tau d\eta. \end{aligned} \tag{31}$$

To prove that such a function $G = G(s,t)$ exists, we use the method of successive approximations. We take as an initial guess

$$G^1(s,t) = -\frac{\lambda t}{6}(2L-t-s) \tag{32}$$

and define the recursive formula as follows,

$$G^{n+1}(s,t) = \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G^n_{sss}(\eta,\xi) + 2G^n_s(\eta,\xi) + \lambda G^n(\eta,\xi)\Big) d\xi d\tau d\eta. \tag{33}$$

Performing some computations, we get for instance

$$G^2(s,t)=\frac{1}{108}\Big\{t^3\big(\lambda-\lambda^2L+\frac{\lambda^2t}{4}\big)\big(2L-t-s\big)+\frac{t^3\lambda^2}{4}\big[(2L-t)^2-s^2\big]\Big\}, \tag{34}$$

and more generally the following formula

$$G^k(s,t)=\sum_{i=1}^{k}\left(a_k^i t^{2k-1}+b_k^i t^{2k}\right)\big[(2L-t)^i-s^i\big], \tag{35}$$

where the coefficients satisfy $b_k^k=0$ and more importantly, there exist positive constants M, B such that, for any $k\geq 1$ and any $(s,t)\in\mathcal{T}_0$

$$\left|G^k(s,t)\right|\leq M\frac{B^k}{(2k)!}(t^{2k-1}+t^{2k}). \tag{36}$$

This implies that the series $\sum_{n=1}^{\infty}G^n(s,t)$ is uniformly convergent in $\mathcal{T}_0$. Therefore the series defines a continuous function $G:\mathcal{T}_0\to\mathbb{R}$

$$G(s,t)=\sum_{n=1}^{\infty}G^n(s,t) \tag{37}$$

and we get a solution of our integral equation. Indeed, we can write

$$\begin{aligned}
G &= G^1+\sum_{n=1}^{\infty}G^{n+1}\\
&= G^1+\frac{1}{6}\sum_{n=1}^{\infty}\int_s^{2L-t}\int_0^t\int_0^\tau\Big(2G^n_{sss}(\eta,\xi)+2G^n_s(\eta,\xi)+\lambda G^n(\eta,\xi)\Big)d\xi d\tau d\eta\\
&= G^1+\frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau\Big(2\sum_{n=1}^{\infty}G^n_{sss}(\eta,\xi)+2\sum_{n=1}^{\infty}G^n_s(\eta,\xi)+\lambda\sum_{n=1}^{\infty}G^n(\eta,\xi)\Big)d\xi d\tau d\eta\\
&= G^1+\frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau\Big(2G_{sss}(\eta,\xi)+2G_s(\eta,\xi)+\lambda G(\eta,\xi)\Big)d\xi d\tau d\eta.
\end{aligned} \tag{38}$$

where we have used that the corresponding series $\sum_{n\geq 1}G^n_s$ and $\sum_{n\geq 1}G^n_{sss}$ are also uniformly convergent.

Once we have found the function $G=G(s,t)$, we get the existence of the kernel $k=k(x,y)$. It is easy to see that the map $\Pi : L^2(0,L)\to L^2(0,L)$, defined by (12), is continuous and consequently we have the existence of a positive constant D_κ such that

$$\|\Pi(f)\|_{L^2(0,L)} \leq D_\kappa \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L). \tag{39}$$

In Figure 2, we plot the gain kernel $k(0,y)$ (see (12)) as a function of $y \in [0,L]$ for different lengths (a) $L = 1$ (non-critical), (b) $L = 2\pi$ (critical) and (c) $L = 2\pi\sqrt{7/3}$ (critical). The kernel functions are defined with $\lambda = 1$. This illustrate the fact that case (a) is easier to stabilizes than case (b), which is easier to stabilizes than case (c). This is due to the location of the corresponding open-loop eigenvalues as shown in Figure 1.

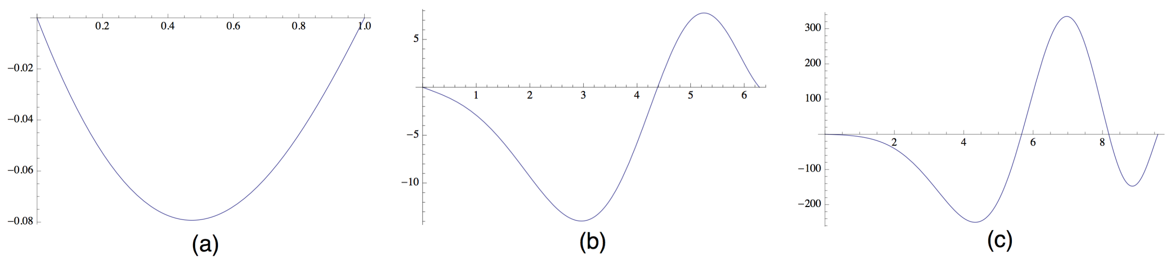

Figure 2. Gain kernel $k(0,y)$ corresponding to $\lambda = 1$ for (a) $L = 1$, (b) $L = 2\pi$, and (c) $L = 2\pi\sqrt{7/3}$, respectively.

IV. Stability of the linear system

We know that the target system (14) is exponentially stable. In order to get the same conclusion for the linear system (11), the method we are applying uses the inverse transformation Π^{-1}. For that, we introduce a kernel function $\ell(x,y)$ which satisfies

$$\begin{cases} \ell_{xxx}(x,y) + \ell_{yyy}(x,y) + \ell_x(x,y) + \ell_y(x,y) = \lambda \ell(x,y), & \text{in } \mathcal{T}, \\ \ell(x,L) = 0, & \text{in } [0,L], \\ \ell(x,x) = 0, & \text{in } [0,L], \\ \ell_x(x,x) = \dfrac{\lambda}{3}(L-x), & \text{in } [0,L], \end{cases} \tag{40}$$

The existence and uniqueness of such a kernel $\ell = \ell(x,y)$ are proven in the same way than for the kernel $k = k(x,y)$ in Section III. Once we have defined $\ell = \ell(x,y)$, it is easy to see that the transformation Π^{-1} is characterized by

$$u(x) = \Pi^{-1}(w(x)) := w(x) + \int_x^L \ell(x,y)w(y)dy. \tag{41}$$

Let us see that $k(x,y)$ and $\ell(x,y)$ are related by the formula

$$\ell(x,y) - k(x,y) = \int_x^y k(x,\eta)\ell(\eta,y)d\eta, \tag{42}$$

which in fact proves that Π^{-1} maps a trajectory of (14) into a trajectory of (11) with control $\kappa(t)$ defined by (13). Indeed, by plugging (41) into (12)and using Fubini's theorem we get, for any $w \in L^2(0,L)$, that for any $x \in [0,L]$

$$\int_x^L (\ell(x,y) - k(x,y))w(y)dy = \int_x^L \int_x^y k(x,\eta)\ell(\eta,y)w(y)d\eta dy, \tag{43}$$

which proves (42) for any $(x,y) \in \mathcal{T}$.

The map $\Pi^{-1} : L^2(0,L) \to L^2(0,L)$ is continuous and therefore we get the existence of a positive constant D_ℓ such that

$$\|\Pi^{-1}(f)\|_{L^2(0,L)} \le D_\ell \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L). \tag{44}$$

Let us prove that system (11)-(13) is exponentially stable. In fact, given $u_0 \in L^2(0,L)$, we define

$$w_0(x) = \Pi(u_0(x)) := u_0(x) - \int_x^L k(x,y)u_0(y)dy. \tag{45}$$

The solution of (14) with initial condition $w(0,x) = w_0(x)$ satisfies (15), i.e.

$$\|w(t,\cdot)\|_{L^2(0,L)} \le e^{-\lambda t}\|w_0(\cdot)\|_{L^2(0,L)}, \quad \forall t \ge 0. \tag{46}$$

Moreover, the solution of (11) is given by $u(t,x) = \Pi^{-1}(w(t,x))$. Thus, from (39),(44) and

(46) we have for any $t \geq 0$

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq D_\ell \|w(t,\cdot)\|_{L^2(0,L)} \leq D_\ell e^{-\lambda t}\|w_0(\cdot)\|_{L^2(0,L)} \leq D_\ell D_k e^{-\lambda t}\|u_0(\cdot)\|_{L^2(0,L)}, \quad (47)$$

which proves the exponential decay at rate λ for system (11) with feedback law (13).

V. Stability of the nonlinear system

Let $u = u(t,x)$ be a solution of the nonlinear equation (6) with the control κ given by (13). Then, $w = \Pi(u(t,x))$ satisfies

$$w_t(t,x) + w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) = -\left(w(t,x) + \int_x^L \ell(x,y)w(t,y)dy\right)\left(w_x(t,x) + \int_x^L \ell_x(x,y)w(t,y)dy\right) \quad (48)$$

with homogeneous boundary conditions

$$w(t,0) = 0, \quad w(t,L) = 0, \quad w_x(t,L) = 0. \quad (49)$$

We multiply (48) by w and integrate in $(0,L)$ to obtain

$$\frac{d}{dt}\int_0^L |w(t,x)|^2 dx \;=\; -|w_x(t,0)|^2 \;-\; 2\lambda\int_0^L |w(t,x)|^2 dx \;-\; 2\int_0^L w(t,x)F(t,x)dx \quad (50)$$

where the term $F = F(t,x)$ is given by

$$F(t,x) = w(t,x)\int_x^L \ell_x(x,y)w(t,y)dy + w_x(t,x)\int_x^L \ell(x,y)w(t,y)dy + \left(\int_x^L \ell(x,y)w(t,y)dy\right)\left(\int_x^L \ell_x(x,y)w(t,y)dy\right). \quad (51)$$

We can prove that there exists a positive constant $C = C(\|\ell\|_{C^1(\mathcal{T})})$ such that

$$\left|2\int_0^L w(t,x)F(t,x)dx\right| \leq C\left(\int_0^L |w(t,x)|^2\right)^{3/2} \quad (52)$$

and therefore, if there exists $t_0 \geq 0$ such that

$$\|w(t_0,\cdot)\|_{L^2(0,L)} \leq \frac{\lambda}{C}, \tag{53}$$

then we obtain

$$\frac{d}{dt}\int_0^L |w(t,x)|^2 dx \leq -\lambda \int_0^L |w(t,x)|^2 dx, \quad \forall t \geq t_0. \tag{54}$$

Thus, we get

$$\|w(t,\cdot)\|_{L^2(0,L)} \leq e^{-\frac{\lambda}{2}t}\|w(0,\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0, \tag{55}$$

provided that

$$\|w(0,\cdot)\|_{L^2(0,L)} \leq \frac{\lambda}{C}. \tag{56}$$

As we did for the linear system, by using the continuity of the transformations Π and Π^{-1} (see (39) and (44)) and (55), we obtain the exponential decay of the nonlinear equation (6). From (56), we have to add a smallness condition on the initial data of system (6). This concludes the proof of Theorem 2.

VI. Some extensions

As mentioned in Remark 3, this method can be applied to stabilize other related KdV systems. In this Section, we focus in the linear control design because the nonlinear part of the argument is the same as in Section V. More precisely, we show the equations that define the kernel functions corresponding to each case.

A. Different boundary conditions

In order to stabilize system

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) = 0, \\ u(t,0) = \kappa(t), \quad u_x(t,L) = 0, \quad u_{xx}(t,L) = 0, \end{cases} \tag{57}$$

we have to consider a kernel $k = k(x,y)$ satisfying on the triangle $\mathcal{T} := \{(x,y) \,/\, x \in [0,L],\ y \in [x,L]\}$ the equation

$$\left\{\begin{array}{rcll} k_{xxx}(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_y(x,y) & = & -\lambda k(x,y), & \text{in } \mathcal{T}, \\ k_{yy}(x,L) & = & 0, & \text{in } [0,L], \\ k(x,x) & = & 0, & \text{in } [0,L], \\ k_x(x,x) & = & \dfrac{\lambda}{3}(L-x), & \text{in } [0,L]. \end{array}\right. \tag{58}$$

The transformation

$$w(x) = \Pi(u(x)) := u(x) - \int_x^L k(x,y)u(y)dy, \tag{59}$$

map the solution $u = u(t,x)$ into the trajectory $w = w(t,x)$, solution of the target system

$$\left\{\begin{array}{l} w_t(t,x) + w_x(t,x) + w_{xxx}(t,x) + \lambda w(t,x) = 0, \\ w(t,0) = 0, \quad w_x(t,L) = 0, \quad w_{xx}(t,L) = 0, \end{array}\right. \tag{60}$$

which is exponentially asymptotically stable for $\lambda > 0$, with a decay at least equals to λ. By making the change of variable

$$t = y - x, \quad s = x + y, \tag{61}$$

and defining $G(s,t) := k(x,y)$, we get that G is solution of

$$\begin{aligned} G(s,t) = & -\frac{\lambda}{6}(2L - s) - 2\int_0^t G_s(s,\tau)d\tau - 2\int_0^t \int_0^\tau G_{ss}(s,\xi)d\xi d\tau \\ & + \frac{1}{6}\int_0^t \int_0^\tau \int_s^{2L-\xi} \Big(-4G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \lambda G(\eta,\xi) - 12G_{sst}(\eta,\xi)\Big) d\xi d\tau d\eta \end{aligned} \tag{62}$$

that can be studied by applying again the method of successive approximations..

B. Non-constant coefficient case

Given two functions $\gamma \in C^1([0,L])$ and $\delta \in C([0,L])$, we consider the non-constant coefficient linear KdV equation

$$\begin{cases} u_t(t,x) + \gamma(x)u_x(t,x) + u_{xxx}(t,x) + \delta(x)u = 0, \\ u(t,0) = \kappa(t), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{cases} \tag{63}$$

and the target system

$$\begin{cases} w_t(t,x) + \gamma(x)w_x(t,x) + w_{xxx}(t,x) + (\lambda + \delta(x))w(t,x) = 0, \\ w(t,0) = 0, \quad w(t,L) = 0, \quad w_x(t,L) = 0, \end{cases} \tag{64}$$

which is exponentially asymptotically stable for λ large enough. In fact,

$$\lambda \geq \max_{x\in[0,L]} |\gamma'(x) - 2\delta(x)|$$

is a sufficient condition to ensure an exponential decay rate equals to $\lambda/2$ for solutions of (64).

In this case, the kernel $k = k(x,y)$ to be considered in order to define the corresponding transformation (59) is the solution of

$$\begin{cases} k_{xxx}(x,y) + k_{yyy}(x,y) + \gamma(x)k_x(x,y) + \gamma(y)k_y(x,y) &= \alpha(x,y)k(x,y), & \text{in } \mathcal{T}, \\ k(x,L) &= 0, & \text{in } [0,L], \\ k(x,x) &= 0, & \text{in } [0,L], \\ k_x(x,x) &= \dfrac{\lambda}{3}(L-x), & \text{in } [0,L]. \end{cases} \tag{65}$$

where α is defined by

$$\alpha(x,y) = -(\lambda + \gamma'(y) + \delta(x) + \delta(y)). \tag{66}$$

By using the change of variable (61), system (65) can be lead to the equation

$$\begin{aligned} G(s,t) = -\frac{\lambda t}{6}(2L-t-s) \\ + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss}(\eta,\xi) + a_1(\eta,\xi)G_s(\eta,\xi) \\ + a_2(\eta,\xi)G_t(\eta,\xi) + \alpha\Big(\frac{\eta-\xi}{2},\frac{\eta+\xi}{2}\Big)G(\eta,\xi)\Big)d\xi d\tau d\eta, \quad (67) \end{aligned}$$

where

$$a_1(\eta,\xi) = \gamma\Big(\frac{\eta+\xi}{2}\Big) + \gamma\Big(\frac{\eta-\xi}{2}\Big) \quad \text{and} \quad a_2(\eta,\xi) = \gamma\Big(\frac{\eta+\xi}{2}\Big) - \gamma\Big(\frac{\eta-\xi}{2}\Big). \quad (68)$$

As before, this equation can be studied by applying the method of successive approximations.

C. A non-zero equilibrium state

We consider $u = u(t,x)$ solution of (6). If instead of linearizing this system around zero, we do that around a non-zero equilibrium state $z = z(x)$ solution of

$$\begin{cases} z'(x) + z'''(x) + z(x)z'(x) = 0, \\ z(0) = U_0, \quad z(L) = 0, \quad z'(L) = 0, \end{cases} \quad (69)$$

with $U_0 \neq 0$, we have to consider the control system

$$\begin{cases} v_t(t,x) + (1+z(x))v_x(t,x) + v_{xxx}(t,x) + z_x(x)v(t,x) = 0, \\ v(t,0) = h(t), \quad v(t,L) = 0, \quad v_x(t,L) = 0, \end{cases} \quad (70)$$

where $v = v(t,x)$ and $h = h(t)$ are the first-order approximation of the state $(u(t,x) - z(x))$ and the control $(\kappa(t) - U_0)$, respectively. Thus, by using the control design in Subsection VI-B for non-constant coefficients, we can locally stabilizes system (6) around a non-zero equilibrium.

VII. Conclusions

We have applied the backstepping method to build some boundary feedback laws, which locally stabilize the Korteweg-de Vries equation posed on a finite interval. Our control acts on the Dirichlet boundary condition at the left hand side of the interval where the system evolves.

The closed-loop system is proven to be locally exponentially stable with a decay rate that can be chosen to be as large as we want. This approach allows to consider the KdV equation (6) with other boundary conditions and low order terms as u_x or u with non-constant coefficients depending on the space variable.

The situation where we act on the right end-point is different. If we consider homogeneous Dirichlet condition on the left and one or two control inputs at the right hand side of the interval, then we are not able to prove the backstepping method works with the transformation (12). Indeed, when imposing

$$w_t + w_x + w_{xxx} + \lambda w = 0 \tag{71}$$

on the target system, we get the following expression at $x = L$ to vanish

$$k(x,L)u_{xx}(t,L) + k(x,L)u(t,L) + k_{yy}(x,L)u(t,L) - k_y(x,L)u_x(t,L). \tag{72}$$

As we do not have to our disposal $u_{xx}(t,L)$, the first term above arises the condition $k(x,L) = 0$. Even if we do not care about the two last terms in (72), in order to keep $w(t,0) = u(t,0) = 0$, we have to impose $k(0,y) = 0$ for any $y \in (0,L)$. With these four boundary restrictions (the other two are on $k(x,x)$), the third order kernel equation satisfied by $k = k(x,y)$ becomes overdetermined. Therefore, it is not clear if such a function $k = k(x,y)$ exists.

A natural idea to deal with controls at $x = L$ is to use the transformation

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \tag{73}$$

instead of (12). However, it is not clear if that approach works. In fact, if we do that, we have to deal now with the extra condition $k_y(x,0) = 0$ for any $x \in (0,L)$. This is due to the fact that when imposing $w_t + w_x + w_{xxx} + \lambda w = 0$ on the target system, we get the extra term $u_x(t,0)k_y(x,0)$ to be cancelled. As previously, this fourth restriction gives an overdetermined kernel equation for $k = k(x,y)$. Moreover, the existence of critical lengths when only one control is considered at the right end-point suggests that either the existence of the kernel or the invertibility of the corresponding map Π should fail for some spatial domains.

Two related problems that remain still open are the boundary global or semi-global stabilization and the output feedback control problem. The boundary global stabilization is hard because it is needed a really nonlinear design as in [14] for the KdV equation with periodic boundary conditions. Concerning the output feedback control problem, we believe it could be solved by applying the backstepping approach in order to built some observers as done in [28], [12] for the heat and the wave equations respectively.

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